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Gauge Invariance Properties of Transition Amplitudes in Gauge Theories. I

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The functional approach developed earlier for scattering theory in quantum field theory makes it possible to make an explicit and complete study of the gauge invariance properties of *transition* amplitudes (not just of the gauge transformations of Green's functions) in covariant and noncovariant gauges. This paper is devoted to the Abelian gauge theory of quantum electrodynamics. Using the powerful technique of functional differentiation and starting from the Coulomb gauge, the gauge invariance property of transition amplitudes, *up to* gauge-dependent scaling factors, is *explicitly* established in arbitrary gauges. The key ingredients in the analysis are the derived exact expression for the vacuum-to-vacuum transition amplitude, introducing in the process arbitrary gauges, and the idea of stimulated emissions by external sources studied earlier.

1. INTRODUCTION

The single most important function that one determines in quantum field theory is the so-called transition amplitude describing the scattering process of an arbitrary number of particles. Recently I have developed (Manoukian, 1987) a functional approach to scattering theory in field theory by deriving an explicit functional differential expression for transition amplitudes. The whole purpose of this approach was to avoid dealing with many of earlier techniques of noncommutativity properties of field operators, avoid solving any field equations of the Yang–Feldman type, avoid solving for Green's functions, avoid dealing with combinatoric problems (such as guessing correct weight factors) associated with the so-called Feynman rules and the old-fashioned Wick's theorem, avoid dealing with creation and annihilation operators, avoid dealing with mass-shell limits as in the LSZ derivation of the reduction formalism, which becomes quite involved for higher spin fields, and, finally, avoid dealing with the often

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quite complicated and usually ambiguous continual (Feynman and Hibbs, 1965; Faddeev, 1981; Manoukian, 1985) path integrals, since the functional differentiation (Schwinger 1951, 1954) provides a solution (Symanzik, 1954; Lam, 1965; Manoukian, 1985) to the latter.

The above functional differential approach turns out to be quite useful for studying gauge invariance properties of transition amplitudes. The gauge transformation properties of Green's functions have been repeatedly studied (e.g., Landau and Khalatnikov, 1956; Johnson and Zumino, 1959; Zumino, 1960; Bialynicki-Birula, 1962; Lukierski, 1963; Johnson, 1964; Abers and Lee, 1973; see also Manoukian, 1986a). The situation with the physically more important function, that is, of the transition amplitude, is very much different. The gauge independence of transition amplitudes, *up to* gauge-dependent scaling factors, is only often stated or just an argument is sketched for its validity (Bjorken and Drell, 1965, p. 198; Abers and Lee, 1973; p. 90; Lee and Zinn-Justin, 1973, p. 1052; Popov and Faddeev, 1972, p. 230), and even then this is just done for the transformation between two particular gauges, such as the Coulomb and the Feynman, or the Coulomb and the Lorentz ones. The purpose of this paper is to consider this problem afresh and give a complete and *explicit* demonstration of this problem. Here we consider only the Abelian gauge theory of quantum electrodynamics; the non-Abelian gauge theories will be considered in a future report. The key ingredients in the analysis are the derived [equation (43)] exact expression for the vacuum-to-vacuum transition amplitude (cf. Manoukian, 1986a), by introducing in the process arbitrary gauges, and the idea of stimulated emissions by external sources studied earlier (Manoukian, 1986b).

2. TRANSITION AMPLITUDES IN ARBITRARY GAUGES

The vacuum-to-vacuum transition amplitude in the *Coulomb* gauge in quantum electrodynamics in the presence of external sources $\mathbf{J}$ (*with* $J^0=0$), η, and $\bar{\eta}$, where the latter are coupled linearly to the 3-vector potential $\mathbf{A}$ and the Fermi fields $\bar{\psi}$ and ψ, respectively, may be inferred directly from Manoukian (1986a) [see in particular equation (2.14)]:

$$\langle 0_+|0_-\rangle = \exp(i\hat{\mathscr{L}}_I)\,\langle 0_+|0_-\rangle_0 \tag{1}$$

where $(j=1,2,3)$

$$\begin{aligned}\hat{\mathscr{L}}_I = \int (dx)\bigg[& e_0(-i)\frac{\delta}{\delta\eta(x)}\gamma^j(-i)\frac{\delta}{\delta\bar{\eta}(x)}(-i)\frac{\delta}{\delta J^j(x)} \\ & +(-i)\frac{e_0^2}{2}(-i)\frac{\delta}{\delta\eta(x)}\gamma^0(-i)\frac{\delta}{\delta\bar{\eta}(x)}\left(\frac{1}{\boldsymbol{\partial}^2}\right)(-i)\frac{\delta}{\delta\eta(x)} \\ & \times\gamma_0(-i)\frac{\delta}{\delta\bar{\eta}(x)}+\delta m(-i)\frac{\delta}{\delta\eta(x)}(-i)\frac{\delta}{\delta\bar{\eta}(x)}\bigg]\end{aligned} \tag{2}$$

$$\langle 0_+|0_-\rangle_0 = \exp\left[i\int (dx)\,(dx')\,\bar{\eta}(x)S_+(x-x')\eta(x')\right]$$
$$\times\exp\left[\frac{i}{2}\int (dx)\,(dx')\,J^i(x)D^C_{ij}(x-x')J^j(x')\right]$$
$$\equiv \exp(i\bar{\eta}S_+\eta)\exp\left(\frac{i}{2}J^iD^C_{ij}J^j\right) \tag{3}$$

and

$$S_+(x-x) = \int \frac{(dp)}{(2\pi)^4}\frac{(-\gamma p+m)}{p^2+m^2-i\varepsilon}e^{ip(x-x')}, \qquad \varepsilon\to +0 \tag{4}$$

expressed in terms of the renormalized mass,

$$D^C_{ij}(x-x') = \int \frac{(dk)}{(2\pi)^4}D^C_{ij}(k)\,e^{ik(x-x')} \tag{5}$$

$$D^C_{ij}(k) = \left(g_{ij}-\frac{k_ik_j}{\mathbf{k}^2}\right)\frac{1}{k^2-i\varepsilon}, \qquad \varepsilon\to +0 \tag{6}$$

$(i, j = 1, 2, 3)$.

We may also write

$$J^i(k)D^C_{ij}(k)J_j(k) = \left(g^{il}-\frac{k^ik^l}{\mathbf{k}^2}\right)J_l(k)D^G_{ij}(k)\left(g^{jm}-\frac{k^jk^m}{\mathbf{k}^2}\right)J_m(k) \tag{7}$$

where

$$D^G_{ij}(k) = \frac{g_{ij}}{k^2-i\varepsilon} - k_ik_jG(k) \tag{8}$$

and $G(k)$ is an arbitrary function. The right-hand side of (7) also may be simply written as

$$\left(g^{ij}-\frac{k^ik^l}{\mathbf{k}^2}\right)J_l(k)\frac{g_{ij}}{k^2-i\varepsilon}\left(g^{jm}-\frac{k^jk^m}{\mathbf{k}^2}\right)J_m(k) \tag{9}$$

For a given 3-vector $\mathbf{k}$ we introduce three unit vectors $\mathbf{e}(\mathbf{k}, 1)$, $\mathbf{e}(\mathbf{k}, 2)$, and $\mathbf{k}/|\mathbf{k}|$, such that

$$\begin{aligned} &k^ie_i(\mathbf{k},\lambda)=0, \qquad \lambda=1,2;\quad i=1,2,3\\ &e^i(\mathbf{k},\lambda)e_i(\mathbf{k},\lambda')^* = \delta_{\lambda\lambda'}, \qquad \lambda,\lambda=1,2\end{aligned} \tag{10}$$

and write the completeness relation:

$$\sum_{\lambda=1,2} e^i(\mathbf{k},\lambda)e^j(\mathbf{k},\lambda)^* + \frac{k^ik^j}{\mathbf{k}^2} = g^{ij} \tag{11}$$

or

$$\sum_{\lambda=1,2} e^i(\mathbf{k}, \lambda) e^j(\mathbf{k}, \lambda)^* = g^{ij} - \frac{k^i k^j}{\mathbf{k}^2} \tag{12}$$

The covariant form of the completeness relation (11) may be also written. To this end, for a given null form-vector k, $k^2 = 0$, one may introduce the orthogonal system (cf. Schwinger, 1970) $e^\mu(k, 1)$, $e^\mu(k, 2)$, $(k - \bar{k})^\mu$, $(k + \bar{k})^\mu$, where $\bar{k}^\mu = (k^0, -\mathbf{k})$:

$$e^\mu(k, \lambda) e_\mu(k, \lambda')^* = \delta_{\lambda\lambda'}, \qquad k_\mu e^\mu(k, \lambda) = 0, \qquad \bar{k}_\mu e^\mu(k, \lambda) = 0 \tag{13}$$

and write the completeness relation

$$\sum_{\lambda=1,2} e^\mu(k, \lambda) e^\nu(k, \lambda^*) + \frac{(k+\bar{k})^\mu (k+\bar{k})^\nu}{2k\bar{k}} - \frac{(k-\bar{k})^\mu (k-\bar{k})^\nu}{2k\bar{k}} = g^{\mu\nu}$$

or

$$\sum_{\lambda=1,2} e^\mu(k, \lambda) e^\nu(k, \lambda)^* + \frac{k^\mu \bar{k}^\nu + k^\nu \bar{k}^\mu}{k\bar{k}} = g^{\mu\nu}$$

The relations in (13) imply the following properties for $e^\mu(k, \lambda)$:

$$e^\mu(k, \lambda) = (0, \mathbf{e}(\mathbf{k}, \lambda)), \qquad k_\mu e^\mu(k, \lambda) = 0, \qquad k^i e_i(\mathbf{k}, \lambda) = 0 \tag{14}$$

with $e_i(\mathbf{k}, \lambda)$ as already given in (10), (12).

Finally we introduce the notation

$$\begin{aligned} j_{\mathbf{k}\lambda} &= \left[\frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\right]^{1/2} e_i(\mathbf{k}, \lambda)^* \left(g^{ij} - \frac{k^i k^j}{\mathbf{k}^2}\right) J_j(k) \\ &= \left[\frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|}\right]^{1/2} e_i(\mathbf{k}, \lambda)^* J^i(k) \end{aligned} \tag{15}$$

and (Manoukian, 1986b)

$$\begin{aligned} \eta^*_{\mathbf{p}\sigma_-} &= (2m\, d\omega_\mathbf{p})^{1/2} \bar{\eta}(p) u(p, \sigma), & \eta_{\mathbf{p}\sigma_-} &= (2m\, d\omega_\mathbf{p})^{1/2} \bar{u}(p, \sigma) \eta(p) \\ \eta^*_{\mathbf{p}\sigma_+} &= (2m\, d\omega_\mathbf{p})^{1/2} \bar{v}(p, \sigma) \eta(-p), & \eta_{\mathbf{p}\sigma_+} &= (2m\, d\omega_\mathbf{p})^{1/2} \bar{\eta}(-p) v(p, \sigma) \end{aligned} \tag{16}$$

where (Schwinger, 1970)

$$d\omega_\mathbf{p} = \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}, \qquad p^0 = (\mathbf{p}^2 + m^2)^{1/2}$$

and the signatures + and − correspond to a particle and an antiparticle, respectively. From equations (3), (7), and (9)-(16), we can then infer directly from Manoukian (1986b) and the functional approach to scattering theory developed earlier (Manoukian, 1987) the explicit expressions for the transition amplitudes.

To this end, we introduce the following notation: $r=(\mathbf{p}, \sigma, \epsilon)$, $\sigma=1, 2$, $\epsilon=\pm$, $\mathbf{p}\in R^3$; $s=(\mathbf{k}, \lambda)$, $\lambda=1, 2$, $\mathbf{k}\in R^3$. We consider first the case $e_0=0$ in the presence of the external sources. The transition amplitude for the scattering of n_1 fermions with values $r_1, \ldots, r_{n_1}$ to n_2 fermions with values $r'_1, \ldots, r'_{n_2}$, and m_1 photons with nonoverlapping values $s_1, \ldots, s_{m_1}$ to m_2 photons with nonoverlapping values $s'_1, \ldots, s'_{m_2}$ is given by, from Manoukian (1986b),

$$\begin{aligned}\langle r'_1, \ldots, r'_{n_2}; s'_1, \ldots, s'_{m_2}|r_1, \ldots, r_{n_1}; s_1, \ldots, s_{m_1}\rangle_0 \\ = (i\eta_{r'_1}) \cdots (i\eta_{r'_{n_2}})(i\eta^*_{r_{n_1}}) \cdots (i\eta^*_{r_1})(ij_{s'_1}) \cdots \\ \times (ij_{s'_{m_2}})(ij^*_{s_1}) \cdots (ij^*_{s_{m_1}})\langle 0_+|0_-\rangle_0\end{aligned} \tag{17}$$

deleting those disconnected parts where at least one particle (electron, positron, or photon) just goes through the process without being scattered (that is, being detected by the corresponding source!).

The exact transition amplitude for the above process for $e_0\neq 0$ is then (Manoukian, 1987)

$$\begin{aligned}\langle r'_1, \ldots, r'_{n_2}; s'_1, \ldots, s'_{m_2}|r_1, \ldots, r_{n_1}; s_1, \ldots, s_{m_1}\rangle \\ = \exp(i\hat{\mathscr{L}}_I)\,\langle r'_1, \ldots, r'_{n_2}; s'_1, \ldots, s'_{m_2}|r_1, \ldots, r_{n_1}; s_1, \ldots, s_{m_1}\rangle_0\end{aligned} \tag{18}$$

by finally setting the external sources equal to zero, where the expression on the extreme right-hand side of equation (18) is *given* in (17), as determined in the Coulomb gauge. Now we make transitions to arbitrary gauges.

We write

$$J_\sigma(q) = \int (dx)\, e^{-iqx} J_\sigma(x), \qquad q^2=0 \tag{19}$$

$$\begin{aligned}\eta(p) &= \int (dx)\, e^{-ipx}\eta(x) \\ &= \int d^3\mathbf{x}\, e^{-ipx} \int (dx')\, \eta(x-x')\chi(x'), \qquad x^0\to+\infty, \quad p^0=(\mathbf{p}^2+m^2)^{1/2}\end{aligned} \tag{20}$$

where

$$\chi(x)=\int \frac{(dp)}{(2\pi)^4}\, e^{ipx}\chi(p) \tag{21}$$

and $\chi(p)$ is an arbitrary, smooth function (infinitely differentiable) of compact support containing the mass shell $p^2+m^2=0$ such that $\chi(p)|_{p^2+m^2=0}=1$.

Since (Manoukian, 1986a)

$$\eta(x)\langle 0_+|0_-\rangle_0=\left(\frac{\gamma\partial}{i}+m\right)(-i)\frac{\delta}{\delta\bar{\eta}(x)}\langle 0_+|0_-\rangle_0 \tag{22}$$

$$\bar{\eta}(x)\langle 0_+|0_-\rangle_0=\left(-\frac{\gamma\partial}{i}+m\right)(-i)\frac{\delta}{\delta\eta(x)}\langle 0_+|0_-\rangle_0 \tag{23}$$

we have

$$\left(g^{ij}-\frac{\partial^i\partial^j}{\partial^2}\right)J_j(x)\langle 0_+|0_-\rangle_0=-\partial_\mu F'^{\mu i}\langle 0_+|0_-\rangle \tag{24}$$

where ($\mu=0,1,2,3$; $i,j=1,2,3$)

$$F'^{\mu\nu}=\left(\partial^\mu(-i)\frac{\delta}{\delta J_\nu}-\partial^\nu(-i)\frac{\delta}{\delta J_\mu}\right) \tag{25}$$

Using also the fact from (10), with $k^2=0$,

$$\begin{aligned}&\int (dx)\, e^{-ikx}e_i(\mathbf{k},\lambda)^*(-\partial_\mu F'^{\mu i})\\ &=\int (dx)\, e^{-ikx}e^i(\mathbf{k},\lambda)^*(-\Box^2)(-i)\frac{\delta}{\delta J^i(x)}\end{aligned} \tag{26}$$

we can rewrite (17), by using finally the *mass shell* restrictions, as

$$U\left(\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}},e^i\frac{\delta}{\delta J^i}\right)\langle 0_+|0_-\rangle_0\bigg|_{\eta=0,\bar{\eta}=0,J=0} \tag{27}$$

corresponding to the elementary $e_0=0$ case, where U is the functional differential operator:

$$\begin{aligned}&\left[\prod_{j=1}^{n_1}(2m\, d\omega_{\mathbf{p}'_j})^{1/2}\hat{A}^j\right]\left[\prod_{j=1}^{n_2}(2m\, d\omega_{\mathbf{p}_j})^{1/2}\hat{B}_j\right]\left[\prod_{j=1}^{m_1}\left(\frac{d^3\mathbf{k}'_j}{(2\pi)^3 2|\mathbf{k}'_j|}\right)^{1/1}\hat{C}_j\right]\\ &\times\left[\prod_{j=1}^{m_2}\left(\frac{d^3\mathbf{k}_j}{(2\pi)^3 2|\mathbf{k}_j|}\right)^{1/2}\hat{D}_j\right]\end{aligned} \tag{28}$$

with

$$\hat{A}_j = \int d^3\mathbf{x}_j \, [\exp(-i\varepsilon j p'_j x_j)][\bar{u}(r'_j)]^{\alpha_j}$$

$$\times \int dX_j \, \chi(X_j) \left(\frac{\gamma\partial^j}{i} + m\right) \frac{\delta}{\delta\bar{\eta}(x_j - X_j)} [v(r'_j)]^{1-\alpha_j}$$

$$\hat{B}_j = \int d^3\mathbf{x}_{j+n_1} \, [\exp(i\varepsilon_{j+n_1} p_j x_{j+n_1})][\bar{v}(r_j)]^{1-\alpha_j}$$

$$\times \int dX_{j+n_1} \, \chi(X_{j+n_1}) \left(-\frac{\gamma\partial^{j+n_1}}{i} + m\right) \frac{\delta}{\delta\eta(x_{j+n_1} - X_{j+n_1})} [u(r_j)]^{\alpha_j}$$

$$\hat{C}_j = \int (dx'_j) \, [\exp(-ik'_j x'_j)] e^m(\mathbf{k}'_j, \lambda'_j)^*(-\Box'_j) \frac{\delta}{\delta J^m(x'_j)}$$

$$\hat{D}_j = \int (dy'_j) \, [\exp(ik_j y'_j)] e^m(\mathbf{k}_j, \lambda_j)(-\Box_j) \frac{\delta}{\delta J^m(y'_y)}$$

where the limits

$$x_1^0 \to \infty, \ldots, x_{n_1}^0 \to \infty, x_{n_1+1}^0 \to -\infty, \ldots, x_{n_1+n_2}^0 \to -\infty$$

are taken independently, $p_j^0 = (\mathbf{p}_j^2 + m^2)^{1/2}$, $p_j'^0 = (p_j'^2 + m^2)^{1/2}$, $q_j^0 = |q_j|$, $q_j'^0 = |\mathbf{q}'_j|$, and $\alpha_j = (1 + \varepsilon_j 1)/2$.

Since U involves only functional differentiations with respect to the sources only and not the sources themselves, we may rewrite the amplitude (18) in question as

$$U\left(\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}, e^i \frac{\delta}{\delta J^i}\right) \exp(i\mathscr{L}'_I) \langle 0_+ | 0_- \rangle_0 \bigg|_{\eta=0, \bar{\eta}=0, J^i=0}$$

$$= U\left(\frac{\delta}{\delta\eta}, \frac{\delta}{\delta\bar{\eta}}, e^i \frac{\delta}{\delta J^i}\right) \langle 0_+ | 0_- \rangle \bigg|_{\eta=0, \bar{\eta}=0, J^i=0} \tag{29}$$

where $\langle 0_+ | 0_- \rangle$ is the exact vacuum-to-vacuum transition amplitude in (1).

We will make the transition to arbitrary gauges. To this end, we rewrite the propagator in (8), by introducing in the process cutoffs μ_0, Λ with $\mu_0 \to 0$, $\Lambda \to \infty$, as

$$D_{ij}^G(k) = \left[\frac{g_{ij}}{k^2 + \mu_0^2 - i\varepsilon} - k_i k_j G(k)\right] \left(\frac{\Lambda^2}{k^2 + \Lambda^2}\right)^a \tag{30}$$

$(i, j = 1, 2, 3)$, by keeping, for convenience, the same notation as in (8), for some suitable integer a such that an expression like (32) together with its first *two* derivatives exist. We consider the functional $\exp(\frac{1}{2}ij^l D^G_{lm} j^m)$, where j^m is arbitrary. It is then readily seen that

$$\exp\left[e_0\epsilon\frac{\partial^m}{\boldsymbol{\partial}^2}\frac{\delta}{\delta j^m(x)}\right]\exp\left(\frac{i}{2}j^l D^G_{lm}j^m\right)$$
$$=\exp\left(\frac{i}{2}j^l D^G_{lm}j^m\right)\exp\left[\frac{ie_0^2}{2}F(0)\right]$$
$$\times\exp\left[ie_0\epsilon\int(dx')\frac{\partial^i_x}{\boldsymbol{\partial}^2}D_{im}(x-x')j^m(x')\right] \qquad (31)$$

where ϵ will be chosen to be ± 1, and where

$$F(x-y)=\frac{\partial^i_x\partial^j_y}{\boldsymbol{\partial}^4}D^G_{ij}(x-y)$$
$$=\int\frac{(dk)}{(2\pi)^4}e^{ik(x-y)}\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^a\left[\frac{1}{\mathbf{k}^2}\frac{1}{k^2+\mu_0^2-i\varepsilon}-G(k)\right] \qquad (32)$$

We also use the identity

$$\exp\left[e_0\epsilon_y\frac{\partial^m}{\boldsymbol{\partial}^2}\frac{\delta}{\delta j^m(x)}\right]\exp\left[ie_0\epsilon_x\int(dx')\frac{\partial^i_x}{\boldsymbol{\partial}^2}D^G_{im}(x-x')j^m(x')\right]$$
$$=\exp\left[ie_0^2\epsilon_x\epsilon_y F(x-y)\right]\exp\left[ie_0\epsilon_x\int(dx')\frac{\partial^i_x}{\boldsymbol{\partial}^2}D^G_{im}(x-x')j^m(x')\right] \qquad (33)$$

Upon using equations (31) and (33) repeatedly, we obtain the following useful identity:

$$\exp\left(\frac{i}{2}j_l D^G_{lm}j^m\right)$$
$$=\left\{\exp\left[-i\frac{e_0^2}{2}F(0)\right]\right\}^n\exp\left[-ie_0^2\sum_{1\le i<j\le n}\epsilon_i\epsilon_j F(x_i-x_j)\right]$$
$$\times\exp\left[ie_0\sum_{i=1}^n\epsilon_i\int(dx')\frac{\partial^l_i}{\boldsymbol{\partial}^2}D^G_{lm}(x_i-x')j^m(x')\right]$$
$$\times\exp\left[e_0\sum_{i=1}^n\epsilon_i\frac{\partial^m_i}{\boldsymbol{\partial}^2}\frac{\delta}{\delta j^m(x_i)}\right]\exp\left(\frac{i}{2}j^l D_{lm}j^m\right) \qquad (34)$$

where n and the space-time points chosen $x_1,\ldots,x_n$ are arbitrary!

From equations (3), (7), (30), and (34), we may write

$$\begin{aligned}
\exp\left(\frac{i}{2}J^i D^C_{ij} J^j\right)
&= \exp\left\{\int (dx)\left[\left(g^{lm}-\frac{\partial^l\partial^m}{\boldsymbol{\partial}^2}\right)J_m(x)\right]\frac{\delta}{\delta j^l(x)}\right\} \\
&\quad\times\exp\left(\frac{i}{2}j_l D^G_{lm} j^m\right)\Bigg|_{j^m=0} \\
&= \exp\left[-i\frac{e_0^2}{2}F(0)\right]\exp\left[-ie_0^2\sum_{1\le i<j\le n}\epsilon_i\epsilon_j F(x_i-x_j)\right] \\
&\quad\times\exp\left\{\int (dx)\left[\left(g^{lm}-\frac{\partial^l\partial^m}{\boldsymbol{\partial}^2}\right)J_m(x)\right]\frac{\delta}{\delta j^l(x)}\right\} \\
&\quad\times\exp\left[e_0\sum_{i=1}\epsilon_i\frac{\partial_i^m}{\boldsymbol{\partial}^2}\frac{\delta}{\delta j_m(x_i)}\right] \\
&\quad\times\exp\left(\frac{i}{2}j^l D^g_{lm} j^m\right)\Bigg|_{j=0}
\end{aligned} \tag{35}$$

where we have used the fact that

$$\int (dx')\frac{\partial_x^s}{\boldsymbol{\partial}^2}D^G_{sl}(x-x')\left(g^{lm}-\frac{\partial'^l\partial'^m}{\boldsymbol{\partial}'^2}\right)J_m(x')=0 \tag{36}$$

for all J_m.

We introduce the external current j^0 and introduce the functional

$$\exp(\tfrac{1}{2}j^\mu D^G_{\mu\nu}j^\nu) \tag{37}$$

(μ, $\nu = 0, 1, 2, 3$), where

$$D^G_{\mu\nu}(k)=\left[\frac{g_{\mu\nu}}{k^2-i\varepsilon}-k_\mu k_\nu G(k)\right]\left(\frac{\Lambda^2}{k^2+\Lambda^2}\right)^a \tag{38}$$

To write an explicit expression for the vacuum-to-vacuum transition amplitude $\langle 0_+|0_-\rangle$ expressed in terms of the covariant propagator (38) in an arbitrary gauge specified by the $G(k)$ term, we use the following identities:

$$\begin{aligned}
&\exp\left[-e_0\int (dx)\frac{\delta}{\delta\eta(x)}\gamma^i\frac{\delta}{\delta\bar{\eta}(x)}\left(g_{il}-\frac{\partial_i\partial_l}{\boldsymbol{\partial}^2}\right)\frac{\delta}{\delta j_l(x)}\right]\exp(ij^0D^G_{0m}j^m) \\
&\quad=\exp(ij^0D^G_{0m}j^m)
\end{aligned} \tag{39}$$

($\mu = 0, 1, 2, 3$) for all G,

$$\exp\left(\frac{i}{2} j^\mu D^G_{\mu\nu} j^\nu\right) \exp\left[-i\frac{e_0^2}{2}\int (dx)\frac{\delta}{\delta\eta(x)}\gamma^0\frac{\delta}{\delta\bar{\eta}(x)}\left(\frac{1}{\boldsymbol{\partial}^2}\right)\right.$$
$$\left.\times\frac{\delta}{\delta\eta(x)}\gamma_0\frac{\delta}{\delta\bar{\eta}(x)}\right]$$
$$= \exp\left[-e_0\int (dx)\frac{\delta}{\delta\eta(x)}\gamma^0\frac{\delta}{\delta\bar{\eta}(x)}\left(g_{0\nu} - \frac{\partial_0\partial^i g_{i\nu}}{\boldsymbol{\partial}^2}\right)\frac{\delta}{\delta j^\nu(x)}\right.$$
$$\times \exp\left(\frac{i}{2} j^\mu D^G_{\mu\nu} j\nu\right) \tag{40}$$

for $\mu_0^2 \to 0$,

$$\exp\left[-ie_0\sum_{j=1}^n \epsilon_1 \partial_0^j \int (dx)\, j^0(x) G(x - x_j)\right]$$
$$\times \exp(ij^0 D^G_{0m} j^m) \exp\left[e_0 \sum_{i=1}^n \epsilon_i \frac{\partial_i^m}{\boldsymbol{\partial}^2}\frac{\delta}{\delta j^m(x_i)}\right]$$
$$= \exp\left[e_0\sum_{i=1}^n \epsilon_i\frac{\partial_i^m}{\boldsymbol{\partial}^2}\frac{\delta}{\delta j^m(x_i)}\right]\exp(ij^0 D^G_{0m} j^m) \tag{41}$$

and finally we use the identity

$$\exp\left[e_0\int (dx)\frac{\delta}{\delta\eta(x)}\gamma^\mu\frac{\delta}{\delta\bar{\eta}(x)}\partial_\mu\left(a^\sigma\frac{\delta}{\delta j\sigma(x)}\right)\right]\exp(i\bar{\eta}S_+\eta)$$
$$= \exp\left\{i\bar{\eta}\left[\exp\left(-e_0 a^\mu\frac{\delta}{\delta j^\mu}\right)\right]S_+\left[\exp\left(e_0 a^\mu\frac{\delta}{\delta j^\mu}\right)\right]\eta\right\} \tag{42}$$

where $a^\mu = (0, \boldsymbol{\partial}/\boldsymbol{\partial}^2)$.

Therefore we may explicitly write the exact vacuum-to-vacuum transition amplitude, starting from the *Coulomb* gauge, as ($\langle 0_+|0_-\rangle = \langle 0_+|0_-\rangle_{\eta,\bar{\eta},J^i}$)

$$\langle 0_+|0_-\rangle = \left\{\exp\left[-i\frac{e_0^2}{2}F(0)\right]\right\}^n \exp\left[-ie_0^2\sum_{1\le i<j\le\eta}\epsilon_i\epsilon_j F(x_i - x_j)\right]$$
$$\times\exp\left\{\int (dx)\left[\left(g^{lm} - \frac{\partial^l\partial^m}{\boldsymbol{\partial}^2}\right)J_l(x)\right]\frac{\delta}{\delta j^m(x)}\right\}$$
$$\times\exp\left[e_0\sum_{i=1}^n\epsilon_i\frac{\partial_i^m}{\boldsymbol{\partial}^2}\frac{\delta}{\delta j^m(x_i)}\right]Z[\rho,\bar{\rho},j]\bigg|_{j^\mu=0} \tag{43}$$

which is independent of n and $x_1, \ldots, x_n$ (!), where

$$Z[\rho, \bar{\rho}, j] = \exp\left\{i \int (dx) \left[e_0(-i)\frac{\delta}{\delta\rho(x)}\gamma^\mu(-i)\frac{\delta}{\delta\bar{\rho}(x)}(-i)\frac{\delta}{\delta j^\mu(x)} + \delta m(-i)\frac{\delta}{\delta\rho(x)}(-i)\frac{\delta}{\delta\bar{\rho}(x)}\right]\right\} \times \exp(i\bar{\rho}S_+\rho)\exp\left(\frac{i}{2}j^\mu D^G_{\mu\nu}j\nu\right)$$
$$\equiv \exp(i\mathscr{L}'_I)\, Z_0[\rho, \bar{\rho}, j] \qquad (44)$$

and

$$\rho(x) = \eta(x)\exp\left[e_0\frac{\partial^m}{\boldsymbol{\partial}^2}\frac{\delta}{\delta j^m(x)}\right] \qquad (45)$$

$F(x-y)$ is defined in (32), $D^G_{\mu\nu}$ in (30), and the terms in (41) dependent on $\partial^j_0 \int j^0(x)G(x-x_j)$ in (41) do not contribute for $j^0=0$. We recall that n and the space-time points $x_1, \ldots, x_n$ are arbitrary. Expression (43) provides a theoretical laboratory for the investigation of all sorts of problems in QED starting from the Coulomb gauge.

We take $n = n_1 + n_2$ in equation (43) with $x_j \to x_j - X_j$ for $j = 1, \ldots, n$, and $\epsilon_j \to -\epsilon_j$ for $j = 1, \ldots, n_2$, in reference to equation (28). Using the property (10) to replace $e^m\,\delta/\delta J^m$ rigorously by $e^m\,\delta/\delta j^m$, and equations (28), (29), and (43)-(45), we may explicitly write the transition amplitude (29) as

$$\left\{\exp\left[-i\frac{e_0^2}{2}F(0)\right]\right\}^n \times \left\{U\left(\frac{\delta}{\delta\rho}, \frac{\delta}{\delta\bar{\rho}}, e^m\frac{\delta}{\delta j^m}\right) \times \exp\left[-ie_0^2 \sum_{1\le i<j\le n} \epsilon_i\epsilon_j F(\cdots)\right]\right\} Z[\rho, \bar{\rho}, j]\Big|_{\rho=0,\bar{\rho}=0,j=0}$$
$$= \left\{\exp\left[-i\frac{e_0^2}{2}F(0)\right]\right\}^n \exp(i\mathscr{L}'_I) \times \left\{U\left(\frac{\delta}{\delta\rho}, \frac{\delta}{\delta\bar{\rho}}, e^m\frac{\delta}{\delta j^m}\right) \times \exp\left[-ie_0^2 \sum_{1\le i<j\le n} \epsilon_i\epsilon_j F(\cdots)\right]\right\} Z_0[\rho, \bar{\rho}, j]\Big|_{\rho=0,\bar{\rho}=0,j=0} \qquad (46)$$

where the points x_i, x_j in $F(\cdots)$ have been identified as described above corresponding to the integration variables X_j in equation (28).

In many practical computations, one chooses $G(k)=0$, leading to the so-called Feynman gauge. Quite generally one chooses $G(k)\sim 1/k^4$ for $k^2\to\infty$, including, in particular, a whole the class of *covariant* (Landau, Yennie, etc.) *gauges*. It is interesting to point out that one may also choose other noncovariant gauges, such as

$$G(k)=\frac{1}{\mathbf{k}^2}\frac{1}{k^2+\mu_0^2-i\varepsilon}+H(k^2) \tag{47}$$

We consider explicitly the expression

$$\left[U\exp -ie_0^2\sum_{1\le i<j\le n}\epsilon_i\epsilon_j F(\cdot\cdot\cdot)\right]Z_0[\rho,\bar{\rho},j] \tag{48}$$

in (46) for $|x_j^0|\to\infty$. By definition, for given values of the integration variables $\mathbf{x}_1,\ldots,\mathbf{x}_n, X_1,\ldots,X_n$,

$$\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j F(\cdot\cdot\cdot)=\sum_{1\le i<j\le n}\varepsilon_i\varepsilon_j F(x_i-x_j-X_i+X_j) \tag{49}$$

Consider the application of $(\gamma\partial^i/i+m)\delta/\delta\bar{\eta}(x_i-X_i)$ in (28) as a contribution to (48). This leads explicitly to

$$\int d^3\mathbf{x}_i\,[\exp(-i\varepsilon_i p_i' x_i)][\bar{u}(r_i')]^{\alpha_i}\int dX_i\,\chi(X_i)[\cdot]Z_0[\rho,\bar{\rho},j][v(r_i')]^{1-\alpha_i} \tag{50}$$

where

$$\begin{aligned}[\cdot]=&\left[i\rho(x_i-X_i)-e_0^2\gamma\partial^i\sum_{i<j\le n}\varepsilon_i\varepsilon_j F(x_i-X_i-x_j+X_j)S(x_i-X_i-\cdot)\rho(\cdot)\right]\\&\times\exp\left[-ie_0^2\sum_{i<j\le n}\varepsilon_i\varepsilon_j F(x_i-X_i-x_j+X_j)\right]\end{aligned} \tag{51}$$

By invoking the Riemann–Lebesgue lemma, we may assert that $F(x_i-X_i-x_j+X_j)$ and its first derivative $\to 0$ for $|x_i^0|\to\infty$ (Adler and Bardeen, 1971), and we conclude that in this limit $[\cdot]\to i\rho(x_i-X_i)$, and (51) be simply replaced by

$$\begin{aligned}&[\bar{u}(r_i')]^{\alpha_i} i\eta(p_i')Z_0[\rho,\bar{\rho},j][v(r_i')]^{1-\alpha_i}\\&=[\bar{u}(r_i')]^{\alpha_i}\int(dx_i)\,[\exp(-i\varepsilon_i p_i' x)]\left(\frac{\gamma\partial^i}{i}+m\right)\\&\times\frac{\delta}{\delta\bar{\eta}(x_i)}Z_0[\rho,\bar{\rho},j][v(r_i')]^{1-\alpha_i}\end{aligned} \tag{52}$$

A similar analysis may be carried out with respect to the other variables x_i.

Finally we use (14) to rewrite $e^m\,\delta/\delta j^m$ in $U(\delta/\delta\rho, \delta/\delta\bar{\rho}, e^m\,\delta/\delta j^m)$ as $e_\mu(g^{\mu\nu}-\partial^\mu\partial^\nu/\partial^2)\,\delta/\delta j^\nu$. All told, we may rewrite the transition in question in (18) as

$$\left\{\exp\left[-i\frac{e_0^2}{2}F(0)\right]\right\}^n \exp(i\mathscr{L}_I')\,\tilde{U}\left(\frac{\delta}{\delta\rho}, \frac{\delta}{\delta\bar{\rho}}, e^\mu\frac{\delta}{\delta j^\mu}\right) Z_0[\rho,\bar{\rho},j]\Big|_{\rho=0,\bar{\rho}=0,j=0} \tag{53}$$

where $\tilde{U}$ may be written as in (28) with $\hat{A}_j$, $\hat{B}_j$, replaced, respectively, by

$$[\bar{u}(r_j')]^{\alpha_j}\int (dX_j)\,[\exp(-i\varepsilon_j p_j' X_j)]\left(\frac{\gamma\partial^j}{i}+m\right)\frac{\delta}{\delta\bar{\eta}(X_j)})\,[v(r_j')]^{1-\alpha_j} \tag{54}$$

$$[\bar{v}(r_j)]^{1-\alpha_j}\int (dX_j)\,[\exp(i\varepsilon_j p_j X_j)]\left(-\frac{\gamma\partial^i}{i}+m\right)\frac{\delta}{\delta\eta(X_j)}[u(r_j)]^{\alpha_j} \tag{55}$$

and $e^{m^*}\,\delta/\delta J^m$ and $e^m\,\delta/\delta J^m$ replaced by

$$e_\mu^*(g^{\mu\nu}-\partial^\mu\partial^\nu/\partial^2)\,\delta/\delta j^\nu(x_j), \qquad e_\mu(g^{\mu\nu}-\partial^\mu\partial^\nu/\partial^2)\,\delta/\delta j^\nu(x_j)$$

in $\hat{C}_j$, $\hat{D}_j$, respectively. The photon propagator in $Z_0[\rho,\bar{\rho},j]$ is defined in (38) [see also (44)].

Equation (53) establishes the equivalence of the Coulomb gauge with the *arbitrary gauges* as defined in (38) for the transition amplitudes, up to gauge-dependent scaling factors such as $[\exp(-ie_0^2/2)\,F(0)]^n$, and is a precise statement of the gauge independence of the so-called *renormalized* transition amplitudes. Finally, when we make the replacement $e_\mu\to e_\mu+q_\mu\Lambda$, then the factor $(g^{\mu\nu}-\partial^\mu\partial^\nu/\partial^2)$ in $e_\mu(g^{\mu\nu}-\partial^\mu\partial^\nu/\partial^2)\,\delta/\delta j^\nu$ ensures the invariance of the latter expression. The situation with regard to non-Abelian gauge theories will be treated in a forthcoming report.

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